Math 4200
Friday October 23
3.1 Infinite sum expressions for analytic functions and their derivatives; introduction to power series.

Announcements: There's a list of Chapter 2.3-2.5 theorems at the end of today's notes, since we'll use them in Chapter 3. We won't go over that list in class. The very last page of notes is our next homework assignment.

Chapter 3: Series representations for analytic functions. Section 3.1: Sequences and series of analytic functions.

Recall a key analysis theorem which we proved and used in our discussion of uniform limits of analytic functions last week, in which we used Morera's theorem to prove that uniform limits of analytic
functions are analytic:

<u>Theorem</u> Let $A \subseteq \mathbb{C}, f_n : A \to \mathbb{C}$ continuous, $n = 1, 2, 3 \ldots$ If $\{f_n\} \to f$ uniformly, then $f$ is continuous. (The same proof would've worked for $A \subseteq \mathbb{R}^k$, $F_n : A \to \mathbb{R}^p$, $\{F_n\} \to F$ uniformly.)

<u>Corollary</u> Let $A \subseteq \mathbb{C}, f_n : A \to \mathbb{C}$ continuous, $n = 1, 2, 3 \ldots$ If $\{f_n\}$ is *uniformly Cauchy*, then there exist a continous limit function $f : A \to \mathbb{C}$, with $\{f_n\} \to f$ uniformly.

<u>Theorem A</u> (essentially from last week, via Morera's Theorem) Let $A \subseteq \mathbb{C}$ open , $f_n : A \to \mathbb{C}$ analytic, and $\{f_n\}$ *uniformly Cauchy*. Then $\exists\, f$ with $\{f_n\} \to f$ uniformly, and $f$ is analytic.

Theorem B (includes statement about convergence of the derivatives, and notes that the convergence doesn't need to be uniform on all of $A$, just locally in $A$.) Let $A \subseteq \mathbb{C}$ open , $f_n : A \to \mathbb{C}$ analytic; $\{f_n(z)\} \to f(z)\ \forall\, z \in A$; $\{f_n\} \to f$ uniformly on each closed disk $\bar{D}(z_0\,; R) \subseteq A$. Then

(1) $f$ is analytic on $A$

(2) Furthermore, the derivatives $f_n{}'(z) \to f'(z)$ and the convergence is uniform on each closed disk $\bar{D}(z_0\,; R) \subseteq A$.

*proof:* (1) follows from Theorem A, applied to each subdisk $D(z_0\,; R)$ with $\bar{D}(z_0\,; R) \subseteq A$.

For (2), Let $\bar{D}(z_0\,; R) \subseteq A$, pick $\rho > 0$ so that also $\bar{D}(z_0\,; R + \rho) \subseteq A$ as well (positive distance lemma). Then for $|z - z_0| \leq R$ use the Cauchy integral formulas for derivatives on the circle of radius $R + \rho$ and compare:

$$f_n{}'(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R + \rho} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R + \rho} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

Recall from analysis the following correspondence between sequences $\{f_n(z)\}$ and series $\sum_{j=1}^{\infty} g_j(z)$:

- Each series $\sum_{j=1}^{\infty} g_j(z)$ corresponds a sequence of partial sums $\{S_n(z)\}$, with $S_n(z) := \sum_{j=1}^{n} g_j(z)$.
- Each sequence $\{f_n(z)\}$ can be rewritten as an infinite series $\sum_{j=1}^{\infty} g_j(z)$ with partial sums $S_n = f_n$ if we define

$$\begin{gathered} g_1(z) := f_1(z) \\ g_2(z) = f_2(z) - f_1(z) \\ \vdots \\ g_n(z) = f_n(z) - f_{n-1}(z). \end{gathered}$$

<u>Def</u> The series $\sum_{j=1}^{\infty} g_j(z)$ *converges uniformly* on the domain $A$ (alternately *at a point* $z \in A$) if and only if the sequence of partial sums $S_n(z) = \sum_{j=1}^{n} g_j(z)$ converges uniformly on $A$ (alternately at a point $z \in A$).

Theorem B (recopied from previous page.) Let $A \subseteq \mathbb{C}$ open, $f_n : A \to \mathbb{C}$ analytic; $\{f_n(z)\} \to f(z)\ \forall\, z \in A$; $\{f_n\} \to f$ uniformly on each closed disk $\bar{\mathrm{D}}(z_0\,; R) \subseteq A$. Then

(1) $f$ is analytic on $A$

(2) Furthermore, the derivatives $f_n{}'(z) \to f'(z)$ and the convergence is uniform on each closed disk $\bar{\mathrm{D}}(z_0\,; R) \subseteq A$.

Theorem B′ (Theorem B restated for series): Let $A \subseteq \mathbb{C}$ open, $g_n : A \to \mathbb{C}$ analytic; $S_n(z) = \sum_{j=1}^{n} g_j(z) \to f(z) = \sum_{j=1}^{\infty} g_j(z)\ \forall\, z \in A$; $\{S_n\} \to f$ uniformly on each closed disk $\bar{\mathrm{D}}(z_0\,; R) \subseteq A$. Then

(1) $f(z) = \sum_{j=1}^{\infty} g_j(z)$ is analytic on $A$

(2) $\frac{d}{dz} f(z) = \frac{d}{dz} \sum_{j=1}^{\infty} g_j(z) = \sum_{j=1}^{\infty} g_j{}'(z)$. Furthermore, the convergence of $\sum_{j=1}^{\infty} g_j{}'(z)$ to $f'(z)$ is uniform on each closed disk $\bar{\mathrm{D}}(z_0\,; R) \subseteq A$. (In other words, we can differentiate the series term by term.)

Def The series $\sum_{j=1}^{\infty} a_j$ converges *absolutely* if and only if $\sum_{j=1}^{\infty} |a_j| < \infty$ .

Theorem: Absolute convergence implies convergence .
*proof:*

There is a useful test for uniform convergence of a series of functions on a domain $A$ - namely a uniform absolute convergence test. It's called the *Weierstrass M test* (maybe $M$ is chosen because of the word M*odulus*), and it's usually covered in Math 3210-3220 in the real-variables context:

Theorem C (Weierstrass M test) Let $\{g_n(z)\}$, $g_n : A \to \mathbb{C}$. If

$$\forall\ n \in \mathbb{N}\ \exists\ M_n \in \mathbb{R} \text{ such that}$$
$$|g_n(z)| \le M_n\ \forall\ z \in A$$

and if

$$\sum_{n=1}^{\infty} M_n < \infty$$

then $\sum_{j=1}^{\infty} g_j(z)$ converges uniformly on $A$. (And in this case, if each $g_n$ is analytic, so is $g(z) = \sum_{j=1}^{\infty} g_j(z)$.)

*proof:*

Examples

(1) Last week we discussed the Zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$. The Weierestrasse $M$ test holds for each half plane with $\mathrm{Re}(z) > 1 + \varepsilon$, for each positive $\varepsilon$, so $\zeta(z)$ is analytic for $\mathrm{Re}(z) > 1$.

(2) Show that

$$\sum_{n=0}^{\infty} z^n$$

Use the Weierstrass M test to show this series converges uniformly on $\mathrm{D}(0, r)$ for any $r < 1$, so converges to an analytic function in all of $\mathrm{D}(0; 1)$. What analytic function is this? (By the way, this is the most important power series in Complex Analysis. :-) )

(3) Show that the series $\sum_{n=0}^{\infty} z^n$ diverges for all $|z| \geq 1$.

(4) Show that the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots.$$

converges uniformly for $|z| \leq R$, so converges to an analytic function $\forall\ z$. Then use the term by term differentiation theorem to show that $f'(z) = f(z)$ and use this and $f(0) = 1$ to identify $f(z)$.

Appendix: Key results of Chapter 2.3-2.5. We'll be using many of these in Chapter 3.

Cauchy's Theorem (Deformation theorem version, section 2.4). Let $f: A \to \mathbb{C}$ analytic.
a) Let $\gamma: [a, b] \to A$ a (piecewise $C^1$) closed contour. If $\gamma$ is homotopic to a point in $A$ as closed curves, then

$$\int_\gamma f(z)\, dz = 0.$$

b) If $\gamma_1, \gamma_2$ are homotopic with fixed endpoints in $A$, then

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

(proofs used the homotopy lemma, which made use of the local antidifferentiation theorem, which used Goursat's rectangle lemma.)

Index The signed number of times a closed contour $\gamma$ winds around $z_0$ can be counted with the index formula

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz$$

(We showed that for any continuous closed contour not containing $z_0$ there's a unique way to measure how the polar angle from $z_0$ to $\gamma(t)$ changes as one traverses $\gamma$, and dividing that total change by $2\pi$ is the definition of the index. We showed that for a piecewise $C^1$ contour, the contour integral expression above computes the same integer.)

Cauchy Integral Formula Let $f: A \to \mathbb{C}$ analytic. Let $\gamma$ be a closed contour homotopic to a point in $A$. Then for $z \notin \gamma$,

$$f(z)\, I(\gamma; z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

(We applied the deformation theorem and local antiderivative theorem with the modified rectangle lemma, for the auxillary function

$$G(\zeta) := \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$
)

Cauchy Integral Formula for derivatives Let $f: A \to \mathbb{C}$ analytic. Then $f$ is infinitely differentiable. And, for $\gamma$ be a closed contour homotopic to a point in $A$ and $z \notin \gamma$,

$$f'(z)\, I(\gamma\,;z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$f^{(n)}(z)\, I(\gamma\,;z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

(We used the fact that if integrands of contour integrals are converging uniformly, then so are the contour integrals, applied to the difference quotients for $f'(z)$; and induction.
)

Estimates: In case $\gamma$ is the index one circle of radius $R$ centered at $z$,

$$\left|f^{(n)}(z)\right| \le \frac{n!}{R^n} \max\left\{\left|f(\zeta)\right| \text{ s.t. } |\zeta - z| = R\right\}.$$

Corollaries Liouville's Theorem: Bounded entire functions are constant.

Theorem Entire functions with moduli that that are bounded by $C\,|z|^n$ with $n \in \mathbb{N}$, for $|z|$ large, are polynomials of degree at most $n$.

Fundamental Theorem of Algebra: Every polynomial $p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ factors as

$$p(z) = (z - \alpha_1)(z - \alpha_2)\ldots.(z - \alpha_n)$$

Morera's Theorem Let $f: A - \mathbb{C}$ continuous and supposed the rectangle lemma holds for every closed rectangle $R \subseteq A$,

$$\int_{\delta A} f(z)\ dz = 0.$$

Then $f$ is analytic in $A$.

(The rectangle lemma means $f$ has local antiderivatives $F$, but these $F$ are twice complex differentiable, so $f$ is complex differentiable.)

Key corollary for Chapter 3: Let $\{f_n\}, f_n : A \to \mathbb{C}$ analytic, $\{f_n\} \to f$ uniformly on $A$. Then $f$ is analytic on $A$.

(We proved that uniform limits of continuous functions are continuous, so that we can compute contour integrals for the limit function $f$. And since each $f_n$ is analytic, and $\{f_n\} \to f$ uniformly, the rectangle lemma hypothesis of Morera's Theorem is satisfied by $f$, so $f$ is analytic.)

Mean value properties

Let $f: A \to \mathbb{C}$ analytic, $\bar{D}(z_0; R) \subseteq A$. Then the value of $f$ at $z_0$ is the average of the values of $f$ on the concentric circle of radius $R$ about $z_0$:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + R\, e^{i\theta}\right) d\theta$$

Let $u: A \to \mathbb{R}$ harmonic and $C^2$, $\bar{D}(z_0; R) \subseteq A$. Then the value of $u$ at $(x_0, y_0)$ is the average of the values of $u$ on the concentric circle of radius $R$ about $z_0$:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(x_0 + R\cos(\theta), y_0 + R\sin(\theta)\right) d\theta$$

Theorem (Maximum modulus principle). Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f: A \to \mathbb{C}$ be analytic, $f: \bar{A} \to \mathbb{C}$ continuous. Then

$$\max_{z \in \bar{A}}\{|f(z)|| = \max_{z \in \delta A}\{|f(z)|\} := M.$$

Furthermore if $\exists\, z_0 \in A$ with $|f(z_0)| = M$, then $f$ is a constant function on $A$.

Theorem (Maximum and minimum principle for harmonic functions). Let $A \subseteq \mathbb{R}^2$ be an open, connected, bounded set. Let $u: A \to \mathbb{R}$ be harmonic and $C^2$, $u: \bar{A} \to \mathbb{R}$ continuous. Then

$$\max_{(x,y) \in \bar{A}}\{u(x,y)\} = \max_{(x,y) \in \delta A}\{u(x,y)\} := M,$$
$$\min_{(x,y) \in \bar{A}}\{u(x,y)\} = \min_{(x,y) \in \delta A}\{u(x,y)\} := m,$$

Furthermore if $\exists\, (x_0, y_0) \in A$ with $u(x_0, y_0) = M$ or $u(x_0, y_0) = m$, then $u$ is a constant function on $A$.

Theorem Let $f: \bar{D}(0;1) \to \bar{D}(0;1)$ be a conformal diffeomorphism of the closed unit disk (i.e. $f$ and $f^{-1}$ are each conformal). Then recording

$$f(0) = z_0,$$

$f$ must be a composition

$$f(z) = g_{z_0}\left(e^{i\theta} z\right)$$

for some choice of $\theta$ and the specific Mobius transformations $g_{z_0}(z)$

$$g_{z_0}(z) = \frac{z_0 + z}{1 + \bar{z}_0 z}.$$

Theorem (Poisson integral formula for the unit disk) Let $u \in C^2(D(0;1)) \cap C(\bar{D}(0;1))$, and let $u$ be harmonic in $D(0;1)$. Then the Poisson integral formula recovers the values of $u$ inside the disk, from the boundary values. It may be expressed equivalently in complex form or real form. For $z_0 = x_0 + i\,y_0 = r\,e^{i\varphi}$ with $|z_0| < 1$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z_0|^2}{\left|z_0 - e^{i\theta}\right|^2} u\left(e^{i\theta}\right) d\theta$$

$$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi} \int_0^{\pi} \frac{1 - r^2}{r^2 - 2r\cos(\theta - \varphi) + 1} u(\cos(\theta), \sin(\theta))\, d\theta$$

Math 4200-001
Week 9-10 concepts and homework
3.1-3.2
Due Friday October 30 at 11:59 p.m.

3.1 4, 12, 13, 14

3.2 2b, 3a, 4 (just for $(1+z)^{\alpha}$), 5c (first four non-zero terms) 7, 13, 14, 18, 19, 20